CANDIDATE NUMBER

SYDNEY GRAMMAR SCHOOL

2018 Trial Examination

FORM VI

MATHEMATICS EXTENSION 1

Friday 17th August 2018

General Instructions

- Reading time — 5 minutes
- Writing time — 2 hours
- Write using black pen.
- Board-approved calculators and templates may be used.

Total — 70 Marks

- All questions may be attempted.

Section I – 10 Marks

- Questions 1–10 are of equal value.
- Record your answers to the multiple choice on the sheet provided.

Section II – 60 Marks

- Questions 11–14 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

Collection

- Write your candidate number on each answer booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Write your candidate number on this question paper and hand it in with your answers.
- Place everything inside the answer booklet for Question Eleven.

Checklist

- SGS booklets — 4 per boy
- Multiple choice answer sheet
- Reference Sheet
- Candidature — 122 boys

Examiner

WJM

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

QUESTION ONE

The point R divides the interval from $A(-4,\ 1)$ to $B(2,\ 7)$ internally in the ratio $5:1$. What are the coordinates of R?

(A) $(6,\ 1)$

(B) $(1,\ 6)$

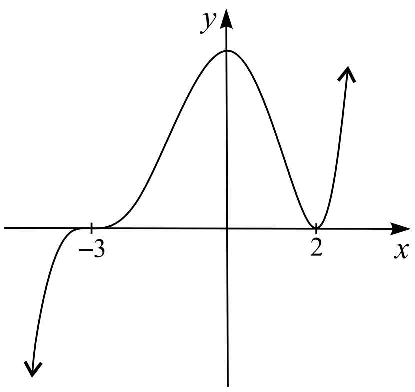
(C) $(-3,\ 2)$

(D) $(2,\ -3)$

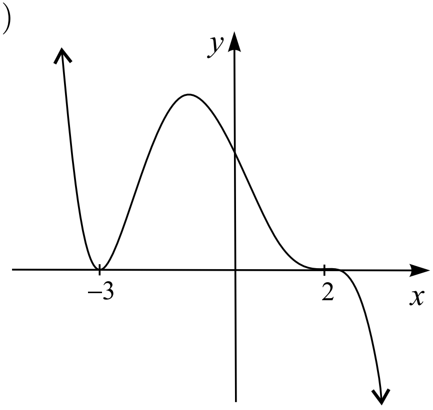
QUESTION TWO

Which of the following diagrams could represent the graph of $y = (x+3)^2(2-x)^3$?

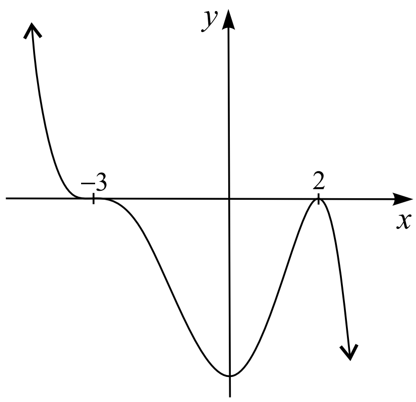
(A)



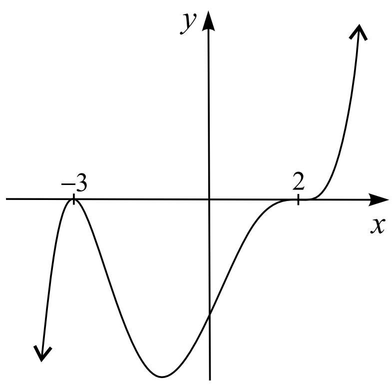
(C)



(B)



(D)



Examination continues next page ...

QUESTION THREE

What is the acute angle between the lines $3x + y + 2 = 0$ and $y = 2 + x$, correct to the nearest degree?

(A) 64°

(B) 27°

(C) 45°

(D) 63°

QUESTION FOUR

What is the derivative of $\sin^{-1} 2x$?

(A) $\dfrac{2}{\sqrt{1-4x^2}}$

(B) $\dfrac{1}{\sqrt{4-x^2}}$

(C) $\dfrac{-2}{\sqrt{1-4x^2}}$

(D) $\dfrac{2}{1+4x^2}$

QUESTION FIVE

A particle is moving along the x-axis. Its velocity $v\,\text{ms}^{-1}$ is given by $v = \sqrt{6x - x^3}$. What is the acceleration of the particle when $x = 2$?

(A) $2\,\text{ms}^{-2}$

(B) $-\dfrac{3}{2}\,\text{ms}^{-2}$

(C) $-6\,\text{ms}^{-2}$

(D) $-3\,\text{ms}^{-2}$

Examination continues overleaf ...

QUESTION SIX

What is the value of k given that $\displaystyle\int_0^k \frac{dx}{9+x^2} = \frac{\pi}{9}$?

(A) $\dfrac{1}{\sqrt{3}}$

(B) $\sqrt{3}$

(C) $3\sqrt{3}$

(D) 3

QUESTION SEVEN

A function is defined by $f(x) = (x+2)^2 + 1$ for $x \le -2$. What is the inverse function of $f(x)$?

(A) $f^{-1}(x) = -2 \pm \sqrt{x-1}$

(B) $f^{-1}(x) = -2 + \sqrt{x-1}$

(C) $f^{-1}(x) = -2 - \sqrt{x-1}$

(D) $f^{-1}(x) = \dfrac{1}{(x+2)^2+1}$

QUESTION EIGHT

A parabola can be represented by the parametric equations $x = 6t,\ y = -3t^2 + 3$. What are the coordinates of the focus?

(A) $(0,\ 3)$

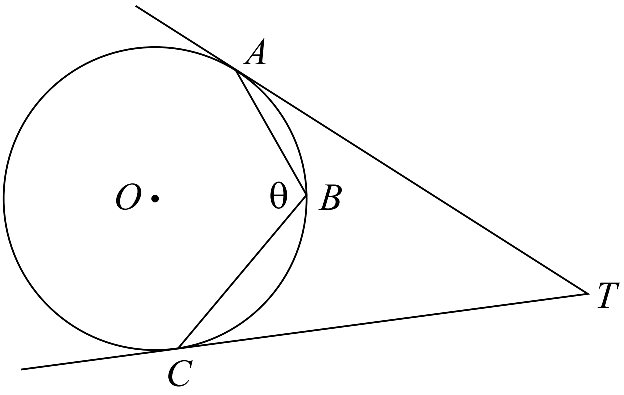
(B) $(0,\ 0)$

(C) $(0,\ -3)$

(D) $(0,\ -6)$

Examination continues next page ...

QUESTION NINE



The diagram above shows a circle through A, B and C, with centre O. Tangents at A and C intersect at T, and $\angle ABC = \theta$.

What is the size of $\angle ATC$ in terms of θ?

(A) $2\theta - 180^\circ$

(B) $180^\circ - \theta$

(C) $\theta - 90^\circ$

(D) $2\theta - 90^\circ$

QUESTION TEN

A polynomial is defined by $P(x) = ax^4 + 2bx^3 + 4cx^2 + 8dx + 16e$ for constants a, b, c, d and e. It is known that $x - 2$ is a factor of $P(x)$, and when $P(x)$ is divided by $x + 2$ the remainder is 32.

What is the value of $b + d$?

(A) 1

(B) -1

(C) 16

(D) -16

End of Section I

Examination continues overleaf ...

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION ELEVEN (15 marks) Use a separate writing booklet. **Marks**

(a) Factorise $8x^3 - y^3$. **1**

(b) Find $\displaystyle\int \frac{dx}{\sqrt{9-x^2}}$. **1**

(c) Find $\displaystyle\int \cos^2 x\, dx$. **2**

(d) The equation $3x^3 - 18x^2 - 25x + 15 = 0$ has roots α, β and γ.

(i) Write down the values of $\alpha + \beta + \gamma$ and $\alpha\beta\gamma$. **1**

(ii) Find the value of $\dfrac{1}{\alpha\beta} + \dfrac{1}{\alpha\gamma} + \dfrac{1}{\beta\gamma}$. **1**

(e) Solve the inequation $\dfrac{6}{x+2} \le 1$. **3**

(f) Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \cos x (1 - \sin x)^2\, dx$ using the substitution $u = 1 - \sin x$. **3**

(g) Find the term independent of x in the expansion of $\left(3x^2 - \dfrac{2}{x}\right)^9$. **3**

Examination continues next page ...

QUESTION TWELVE (15 marks) Use a separate writing booklet. **Marks**

(a) (i) State the domain and range of $y = \sin^{-1}(x+1) + \dfrac{\pi}{2}$. **2**

(ii) Hence sketch the graph of $y = \sin^{-1}(x+1) + \dfrac{\pi}{2}$. **1**

(b) (i) Show that $\operatorname{cosec} 2\alpha - \cot 2\alpha = \tan\alpha$. **2**

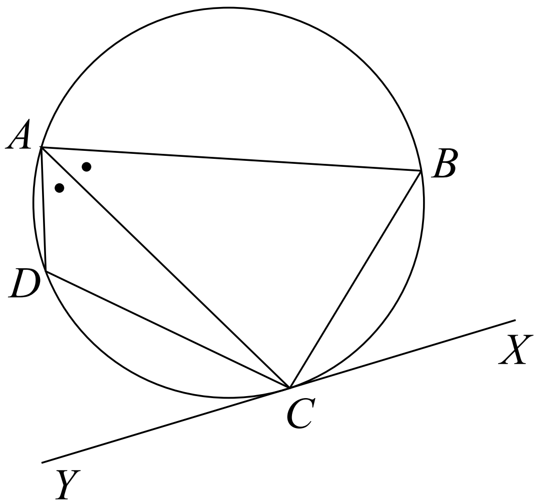
(ii) Hence find the exact value of $\tan 15^\circ$. **1**

(c) A continuous function is defined by $f(x) = \tan\frac{x}{2} - x$ for $-\pi < x < \pi$.

(i) Show that $f(x)$ has a root between $x = 2$ and $x = 3$. **1**

(ii) Taking $x = 2{\cdot}5$ as an initial approximation, use Newton's method once to find a better approximation of the root. Give the value of your approximation correct to two decimal places. **2**

(d) **2**

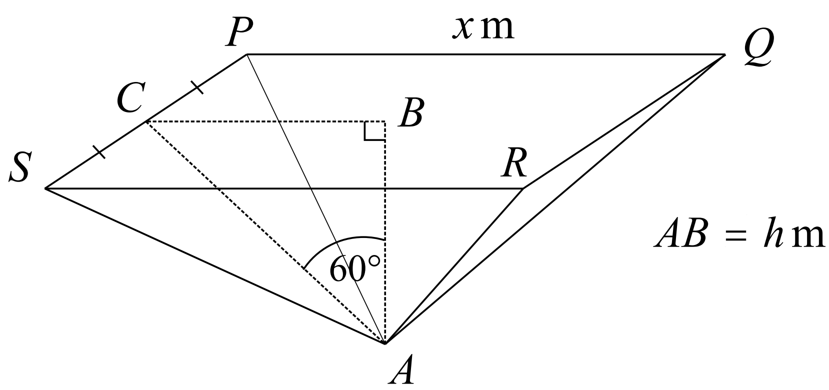


The diagram above shows the cyclic quadrilateral $ABCD$, such that AC bisects $\angle BAD$. The line XY is a tangent to the circle at C.

Show that BD is parallel to XY.

QUESTION TWELVE (Continued)

(e)



The diagram above shows a square pyramid with base $PQRS$. The centre of the base is the point B, and the perpendicular height AB of the pyramid is h m. The side length of the square base is x m, and the slant height AC of the pyramid makes an angle of 60° to the vertical, as shown in the diagram.

(i) Show that $x = 2h\sqrt{3}$. **[1]**

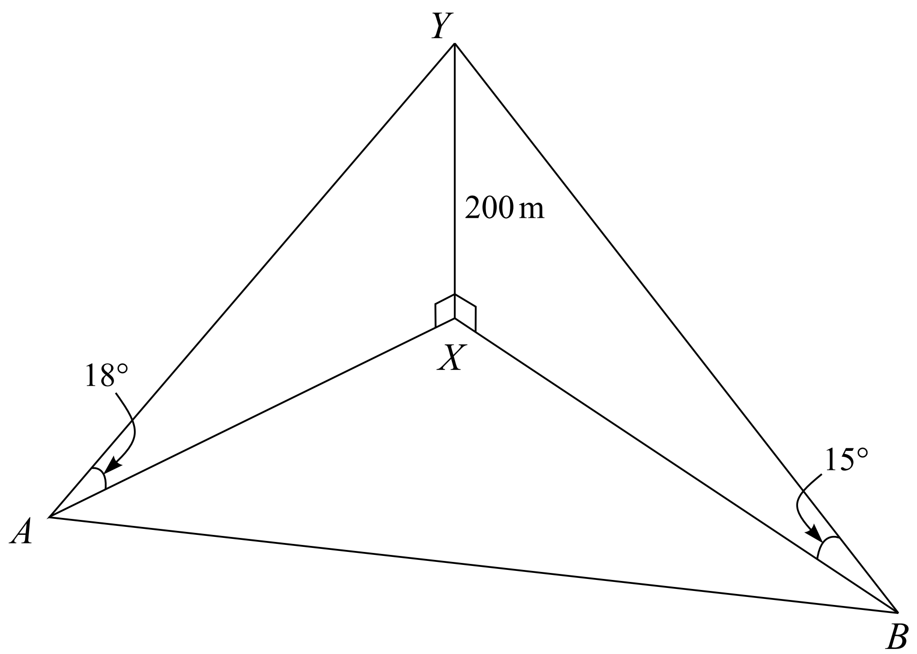
(ii) Hence show that $h = \left(\dfrac{V}{4}\right)^{\frac{1}{3}}$, where V is the volume of the pyramid in cubic metres. **[1]**

(iii) The pyramid is initially empty, and is filled with water at a constant rate of $0{\cdot}5\,\text{m}^3/\text{min}$. Find that rate at which the height of the water is increasing after it has been filled for 8 minutes. **[2]**

Examination continues next page ...

QUESTION THIRTEEN (15 marks) Use a separate writing booklet. **Marks**

(a)



Abby and Bob are bushwalking on flat ground. They can both see the same mountain, which has a height of 200 m. From Abby's position A, the mountain has a bearing of 035°T, and the top of the mountain has an angle of elevation of 18°. From Bob's position B, the mountain has a bearing of 335°T, and the top of the mountain has an angle of elevation of 15°. Let the base and the top of the mountain be the points X and Y respectively.

(i) Show that $\angle AXB = 60°$. **1**

(ii) Find the distance from Abby to Bob. Give your answer correct to the nearest metre. **2**

(b) The roots of $x^3 + 9x^2 + kx - 216 = 0$ form a geometric progression. Find the roots. **3**

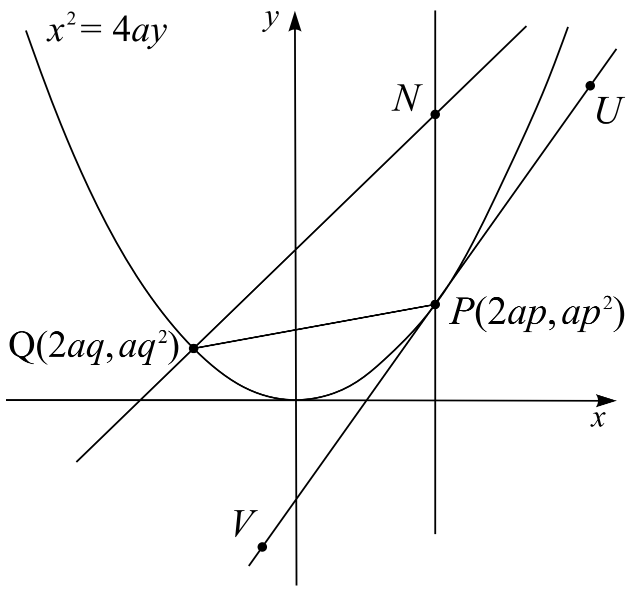
(c) A particle moves such that its displacement in metres after t seconds is given by $x = \sqrt{3}\sin 2t - \cos 2t - 1$.

(i) Prove that the particle is moving in simple harmonic motion by showing that $\ddot{x} = -4(x + 1)$. **2**

(ii) Find the first time that the particle is the furthest from the origin. **3**

QUESTION THIRTEEN (Continued)

(d)



The diagram above shows the parabola $x^2 = 4ay$ with a tangent at the point $P(2ap,\, ap^2)$ and a normal at the point $Q(2aq,\, aq^2)$. The vertical line through P has been constructed, and it intersects the normal through Q at N. The points U and V lie on the tangent to the parabola at P, as shown.

The chord PQ has equation $x(p+q) - 2y - 2apq = 0$. (Do NOT prove this.)

(i) Find the gradient of the tangent to the parabola at P. **[1]**

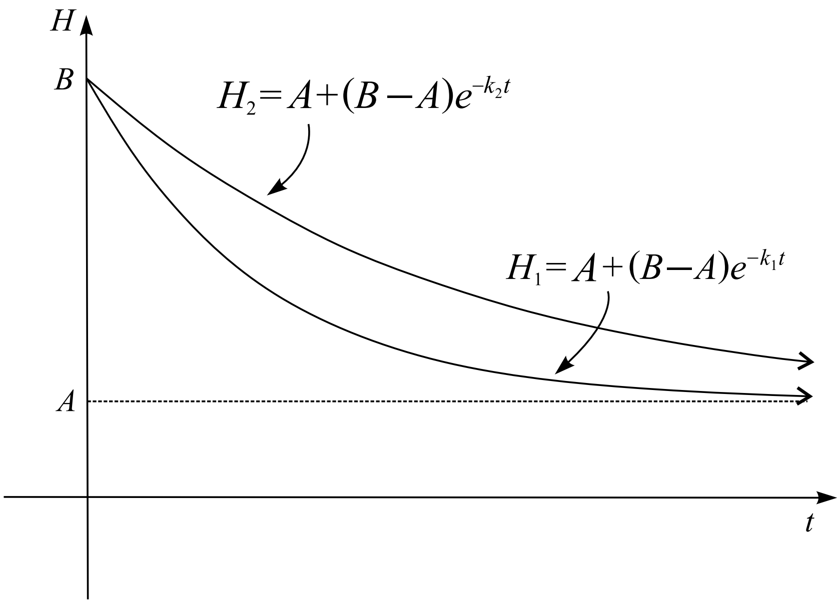
(ii) Show that if PQ is a focal chord, then the normal to the parabola at Q is parallel to the tangent to the parabola at P. **[1]**

(iii) The circle through the points N, Q and P is constructed. Use the reflection property of the parabola (that is, that $\angle QPV = \angle NPU$) to show that if PQ is a focal chord, the parabola and the circle have a common tangent at P. **[2]**

Examination continues next page ...

QUESTION FOURTEEN (15 marks) Use a separate writing booklet. **Marks**

(a)



A liquid is heated to B degrees Celsius before being poured into two different containers, container one and container two. The containers are then placed in a room that is A degrees Celsius, where $A < B$. The temperature H of each liquid after t minutes is graphed above. The temperatures of the liquids in container one and container two after t minutes are given by

$$H_1 = A + (B - A)e^{-k_1 t}$$

$$H_2 = A + (B - A)e^{-k_2 t}$$

respectively, where k_1 and k_2 are positive constants.

It is known that it takes the liquid in container two twice as long to cool to a given temperature as it does for the liquid in container one.

(i) Show that $k_1 = 2k_2$. **2**

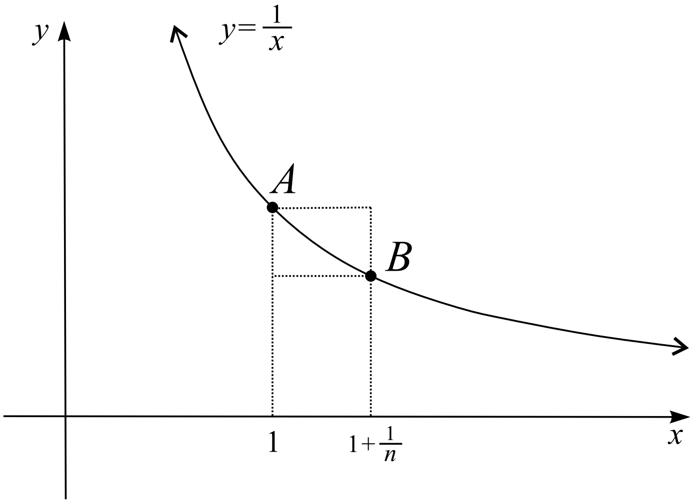
(ii) Find the largest difference in temperatures between the liquids in the two containers. **3**

QUESTION FOURTEEN (Continued)

(b) Use mathematical induction to show that, for any integer $n \geq 0$, **3**

$$\lim_{p\to\infty} \frac{p!}{(p-n)!p^n} = 1 \ .$$

(c)



The diagram above shows the points A and B on the curve $y = \dfrac{1}{x}$ with x-coordinates 1 and $\left(1+\frac{1}{n}\right)$ respectively, where $n > 0$.

(i) By referring to the diagram, explain why **2**

$$\frac{1}{n+1} \leq \int_1^{1+\frac{1}{n}} \frac{dx}{x} \leq \frac{1}{n} \ .$$

(ii) Prove that **2**

$$e^{\frac{n}{n+1}} \leq \left(1+\tfrac{1}{n}\right)^n \leq e \ .$$

(iii) Deduce that $\displaystyle\lim_{n\to\infty} \left(1+\tfrac{1}{n}\right)^n = e$. **1**

(d) Let $a_{n,k}$ be the term containing x^k in the expansion of $(1+x)^n$, where $0 < x < 1$. **2**

It is known that $\displaystyle\lim_{n\to\infty} \sum_{k=0}^{n} a_{n,k} = \sum_{k=0}^{\infty} \lim_{n\to\infty} a_{n,k}$.

Use your answers to (b) and (c)(iii) to show that

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots \ .$$

End of Section II

END OF EXAMINATION

Extension 1 Maths - Trial 2018

Multiple Choice

(1) $A(-4, 1)$, $B(2, 7)$

$5 : 1$

$x = \frac{-4 + 10}{6}$ $\quad$ $y = \frac{1 + 35}{6}$

$= 1$ $\quad$ $= 6$

$\Rightarrow B$

(2) C

(3) $3x + y + 2 = 0$ $\quad$ $y = 2 + x$

$m_1 = -3$ $\quad$ $m_2 = 1$

$\tan\theta = \left| \frac{-3 - 1}{1 - (-3) \times 1} \right|$

$= \left| \frac{-4}{2} \right|$

$\theta = \tan^{-1} 2$

$\doteqdot 63°$ $\quad \Rightarrow D$

(4) $\frac{d}{dx}(\sin^{-1} 2x) = \frac{1}{\sqrt{1-(2x)^2}} \times 2$

$= \frac{2}{\sqrt{1-4x^2}}$ $\quad \Rightarrow A$

(5) $v = \sqrt{6x - x^3}$

$\frac{1}{2}v^2 = \frac{1}{2}(6x - x^3)$

$\frac{d}{dx}(\frac{1}{2}v^2) = \frac{1}{2}(6 - 3x^2)$

When $x = 2$: $\quad \ddot{x} = \frac{1}{2}(6 - 3 \times 2^2)$

$= -3 \text{ ms}^{-2}$ $\quad \Rightarrow D$

(6) $\displaystyle\int_0^k \frac{dx}{9+x^2} = \left[\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\right]_0^k$

$= \frac{1}{3}\left(\tan^{-1}\frac{k}{3} - 0\right)$

$\frac{1}{3}\tan^{-1}\frac{k}{3} = \frac{\pi}{9}$

$\tan^{-1}\frac{k}{3} = \frac{\pi}{3}$

$\frac{k}{3} = \tan\frac{\pi}{3}$

$= \sqrt{3}$

$\therefore\ k = 3\sqrt{3}$

$\Rightarrow$ C

(7) $x = (y+2)^2 + 1$

$(y+2)^2 = x - 1$

$y + 2 = -\sqrt{x-1}$ since range will be $y \leq -2$

$y = -2-\sqrt{x-1}$

$\Rightarrow$ C

(8) $x = 6t,\quad y = -3t^2+3$

$t = \frac{x}{6}$ $\quad\therefore\ y = -3\left(\frac{x}{6}\right)^2 + 3$

$y = -\frac{x^2}{12} + 3$

$12y = -x^2 + 36$

$x^2 = -12(y-3)$

$\therefore$ Focal length $= 3$, Shifted up by 3, concave down.

$\therefore$ Focus is $(0,0)$ $\Rightarrow$ B

Alternatively: $x^2 = 4ay$: $(2at, at^2)$

$a = 3$ gives $(6t, 3t^2)$

So $(6t, -3t^2+3)$ is the standard parabola with vertex $(0,0)$ and focus $(0,3)$ reflected in y-axis & shifted up by 3.

(9) Reflex $\angle AOC = 2\theta$ (angle at centre = 2 x angle at circumference)

$\therefore \angle AOC = 360° - 2\theta$ (angles in a revolution)

$\angle OAT = \angle OCT = 90°$ (angle between radius and tangent)

By angle sum of ATCO:

$$\angle ATC + 90° + 90° + 360° - 2\theta = 360°$$

$$\angle ATC = 2\theta - 180°$$

$\Rightarrow$ A

(10) $P(2) = 0$

$$\therefore 16a + 16b + 16c + 16d + 16e = 0$$

$$a + b + c + d + e = 0 \quad (1)$$

$P(-2) = 32$:

$$16a - 16b + 16c - 16d + 16e = 32$$

$$a - b + c - d + e = 2 \quad (2)$$

(1) + (2):

$$2a + 2c + 2e = 2$$

$$a + c + e = 1$$

sub. into (1): $b + d + (a + c + e) = 0$

$$\therefore b + d = -1$$

$\Rightarrow$ B

Question 11

(a) $(2x-y)(4x^2+2xy+y^2)$

(b) $\displaystyle\int \frac{dx}{\sqrt{9-x^2}} = \sin^{-1}\left(\frac{x}{3}\right) + c$

(c) $\displaystyle\int \cos^2 x \, dx = \frac{1}{2}\int (1+\cos 2x)\,dx$

$\displaystyle = \frac{1}{2}\left[x + \frac{1}{2}\sin 2x\right] + c$

$\displaystyle = \frac{1}{2}x + \frac{1}{4}\sin 2x + c$

(d) (i) $\alpha+\beta+\gamma = -\frac{-18}{3}$

$= 6$

$\alpha\beta\gamma = -\frac{15}{3}$

$= -5$

(ii) $\displaystyle \frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\beta\gamma} = \frac{\alpha+\beta+\gamma}{\alpha\beta\gamma}$

$\displaystyle = -\frac{6}{5}$

(e) $\displaystyle \frac{6}{x+2} \le 1 \qquad x \ne -2$

$6(x+2) \le (x+2)^2$

$(x+2)^2 - 6(x+2) \ge 0$

$(x+2)\left[(x+2)-6\right] \ge 0$

$(x+2)(x-4) \ge 0$

$\therefore \quad x < -2, \quad x \ge 4$

(obtaining -2 and 4)

(f) $\int_0^{\frac{\pi}{2}} \cos x(1-\sin x)^2\,dx$

$u = 1-\sin x$

$du = -\cos x\,dx$

$x=0,\quad u=1$

$x=\frac{\pi}{2},\quad u=0$

$= -\int_0^{\frac{\pi}{2}} -\cos x(1-\sin x)^2\,dx$

$= -\int_1^0 u^2\,du$

$= \int_0^1 u^2\,du$

$= \left[\frac{u^3}{3}\right]_0^1$

$= \frac{1}{3}$

(g) General term: ${}^9C_r\,(3x^2)^r\left(-\frac{2}{x}\right)^{9-r}$

$= {}^9C_r\,3^r(-2)^{9-r}x^{2r}x^{r-9}$

$= {}^9C_r\,3^r(-2)^{9-r}x^{3r-9}$

Constant when $3r-9=0$

$r=3$

$\therefore$ Constant term is ${}^9C_3\,3^3(-2)^6$

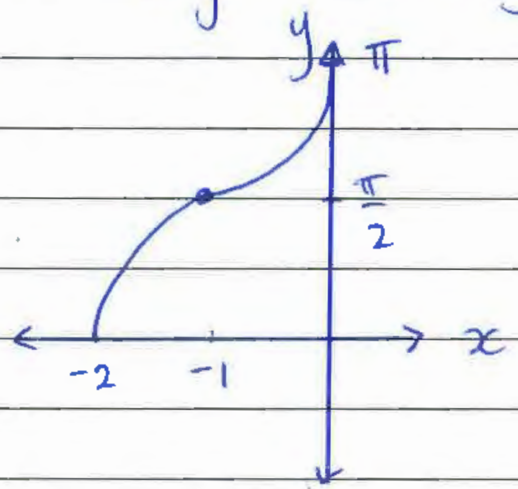
$= 145152$

Question 12

(a)(i) Domain: $-2 \le x \le 0$

Range: $0 \le y \le \pi$

(ii)



(b)(i) $\operatorname{cosec} 2\alpha - \cot 2\alpha = \tan\alpha$

~~LHS = cosec 2α − cot 2α~~

$$= \frac{1}{\sin 2\alpha} - \frac{1}{\tan 2\alpha}$$

$$= \frac{1}{2\sin\alpha\cos\alpha} - \frac{1-\tan^2\alpha}{2\tan\alpha}$$

$$= \frac{1}{2\sin\alpha\cos\alpha} - \frac{(1-\tan^2\alpha)\cos^2\alpha}{2\sin\alpha\cos\alpha}$$

$$= \frac{1-(\cos^2\alpha - \sin^2\alpha)}{2\sin\alpha\cos\alpha}$$

$$= \frac{1-\cos^2\alpha + \sin^2\alpha}{2\sin\alpha\cos\alpha}$$

$$= \frac{2\sin^2\alpha}{2\sin\alpha\cos\alpha}$$

$$= \tan\alpha$$

$$= RHS$$

Alternatively: Let $t = \tan\alpha$

then $\sin 2\alpha = \dfrac{2t}{1+t^2}$, $\tan\alpha = \dfrac{2t}{1-t^2}$

$$\therefore \quad \operatorname{cosec} 2\alpha - \cot 2\alpha = \frac{1+t^2}{2t} - \frac{1-t^2}{2t}$$

$$= \frac{2t^2}{2t}$$

$$= t$$

$$= \tan\alpha$$

(ii) $$\tan 15^\circ = \operatorname{cosec} 30^\circ - \cot 30^\circ$$

$$= \frac{1}{\sin 30^\circ} - \frac{1}{\tan 30^\circ}$$

$$= 2 - \sqrt{3}$$

(c) (i) $$f(2) = \tan\left(\tfrac{2}{2}\right) - 2$$

$$\doteqdot -0.44\ldots < 0$$

$$f(3) = \tan\left(\tfrac{3}{2}\right) - 3$$

$$\doteqdot 11.10\ldots > 0$$

Since $f(x)$ changes sign between $x = 2$ and $x = 3$, and $f(x)$ is continuous over $2 \le x \le 3$

(ii) $$f(x) = \tan\left(\tfrac{x}{2}\right) - x$$

$$f'(x) = \tfrac{1}{2}\sec^2\left(\tfrac{x}{2}\right) - 1$$

$$x_2 = 2.5 - \frac{\tan\left(\frac{2.5}{2}\right) - 2.5}{\frac{1}{2\cos^2\left(\frac{2.5}{2}\right)} - 1}$$

$$\doteqdot 2.37 \quad (2 \text{ decimal places})$$

(d) $\angle DAC = \angle DBC$ (angles on circumference subtended by arc DC)

$\angle BAC = \angle BCX$ (angle between tangent and chord = angle in alternate segment)

Since $\angle DAC = \angle BAC$,

$\angle DBC = \angle BCX$

$\therefore DB \parallel YX$ (alternate angles are equal)

(e) (i) $\tan 60° = \dfrac{\frac{1}{2}x}{h}$

$\sqrt{3} = \dfrac{\frac{1}{2}x}{h}$

$x = 2h\sqrt{3}$

(ii) $V = \frac{1}{3}x^2h$

$= \frac{1}{3}(2h\sqrt{3})^2 \times h$

$V = 4h^3$

$h^3 = \frac{1}{4}V$

$h = (\frac{1}{4}V)^{\frac{1}{3}}$

(ii) $\dfrac{dh}{dt} = \dfrac{dh}{dV} \times \dfrac{dV}{dt}$

$\dfrac{dV}{dt} = 0.5$

$\dfrac{dh}{dV} = \frac{1}{3}(\frac{1}{4}V)^{-\frac{2}{3}} \times \frac{1}{4}$

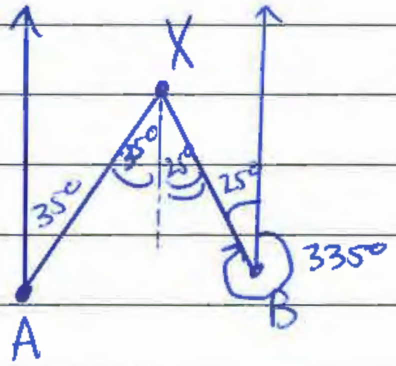
$= \dfrac{1}{12(\frac{1}{4}V)^{2/3}}$

After 8 mins, $V = 8 \times 0.5 = 4$

$\therefore \dfrac{dh}{dt} = \dfrac{1}{12(1)^{2/3}} \times 0.5 = \dfrac{1}{24}$ m/min

Question 13

(a) (i)



$\angle AXB = 35° + 25°$

$= 60°$

(ii) $\tan 18° = \dfrac{200}{AX}$

$AX = \dfrac{200}{\tan 18°}$

Similarly $BX = \dfrac{200}{\tan 15°}$

(both)

By cosine rule:

$$AB^2 = \left(\frac{200}{\tan 18°}\right)^2 + \left(\frac{200}{\tan 15°}\right)^2 - 2 \times \frac{200}{\tan 18°} \times \frac{200}{\tan 15°} \times \cos 60°$$

$= 476570.71\ldots$

$AB \doteqdot 690.34$

$\doteqdot 690$ m (nearest metre)

(b) Let the roots be $\frac{\alpha}{r}, \alpha, \alpha r$

Product of roots: $\frac{\alpha}{r} \times \alpha \times \alpha r = -\frac{-216}{1}$

$\alpha^3 = 216$

$\alpha = 6$

Sum of roots: $\frac{6}{r} + 6 + 6r = -\frac{9}{1}$

$\frac{2}{r} + 2 + 2r = -3$

$2r^2 + 5r + 2 = 0$

$(2r+1)(r+2) = 0$

$r = -\frac{1}{2}$ or $r = -2$

$\therefore$ roots are $-3, 6, -12$

Alternatively:

Let the roots be $\alpha, \alpha r, \alpha r^2$

Product of roots: $\alpha \cdot \alpha r \cdot \alpha r^2 = 216$

$$\alpha^3 r^3 = 216$$

$$\alpha r = 6$$

Sum of roots: $\alpha + \alpha r + \alpha r^2 = -9$

$$\alpha + 6 + 6r = -9$$

$$\alpha + 6r = -15 \quad (*)$$

Two at a time: $\alpha \cdot \alpha r + \alpha \cdot \alpha r^2 + \alpha r \cdot \alpha r^2 = k$

$$\alpha^2 r + \alpha^2 r^2 + \alpha^2 r^3 = k$$

$\alpha r = 6$: $6\alpha + 36 + 36r = k$

$$6(\alpha + 6r) + 36 = k$$

$$6 \times -15 + 36 = k \quad (\text{from } *)$$

$$k = -54$$

$\therefore$ Polynomial is $x^3 + 9x^2 - 54x - 216 = 0$ with 6 as one root.

$$(x-6)(x^2 + 15x + 36) = 0$$

$$(x-6)(x+3)(x+12) = 0$$

$\therefore$ roots are $x = -3$, $x = 6$, $x = -12$

(c) (i) $x = \sqrt{3}\sin 2t - \cos 2t - 1$

$\dot{x} = 2\sqrt{3}\cos 2t + 2\sin 2t$

$\ddot{x} = -4\sqrt{3}\sin 2t + 4\cos 2t$

$= -4(\sqrt{3}\sin 2t - \cos 2t)$

$= -4(x+1)$

Alternatively:

Let $\sqrt{3}\sin 2t - \cos 2t = R\sin(2t+\alpha)$

$= R\sin 2t\cos\alpha + R\cos 2t\sin\alpha$

$\Rightarrow \quad R\cos\alpha = \sqrt{3} \quad ①$

$R\sin\alpha = -1 \quad ②$

$①^2 + ②^2$: $\quad R^2\cos^2\alpha + R^2\sin^2\alpha = (\sqrt{3})^2 + (-1)^2$

$R^2 = 4$

$R = 2$ taking $R > 0$.

$\left.\begin{array}{l} 2\cos\alpha = \sqrt{3} \\ 2\sin\alpha = -1 \end{array}\right\}$ α 4th quadrant.

$\cancel{\sin\alpha = -\frac{1}{2}}$

$\alpha = -\frac{\pi}{6}$ (or $\frac{11\pi}{6}, \ldots$)

$\therefore \quad x = 2\sin(2t - \frac{\pi}{6}) - 1$

$\dot{x} = 4\cos(2t - \frac{\pi}{6})$

$\ddot{x} = -8\sin(2t - \frac{\pi}{6})$

$= -4 \times 2\sin(2t - \frac{\pi}{6})$

$= -4(x+1)$

(ii) Centre of motion: $x = -1$

Furthest from the origin will be in the negative direction, when $\dot{x} = 0$:

When $t = 0$, $\dot{x} = +2\sqrt{3}$

i.e. particle is initially moving in positive direction, so will be "furthest from the origin" the second time that it comes to rest.

$\dot{x} = 0$: $\quad 2\sqrt{3}\cos 2t + 2\sin 2t = 0$

$$\tan 2t = -\sqrt{3}$$

$$2t = \pi - \frac{\pi}{3},\ 2\pi - \frac{\pi}{3},\ 3\pi - \frac{\pi}{3}, \ldots$$

$$= \frac{2\pi}{3},\ \frac{5\pi}{3},\ \frac{8\pi}{3}, \ldots$$

$$t = \frac{\pi}{3},\ \frac{5\pi}{6},\ \frac{4\pi}{3}, \ldots$$

$\therefore$ particle is furthest from the origin when $t = \frac{5\pi}{6}$

(d)(i) $y = \frac{x^2}{4a}$

$y' = \frac{x}{2a}$

when $x = 2ap$, $y' = p$

(ii) Similarly, at $Q(2aq, aq^2)$, $y' = q$

$\therefore$ m normal at $Q = -\frac{1}{q}$

If PQ is a focal chord, $(0,a)$ lies on $x(p+q) - 2y - 2apq = 0$

$\therefore -2a - 2apq = 0$

$pq = -1$

$p = -\frac{1}{q}$

$\therefore$ tangent at P is parallel to normal at Q.

(iii) From (ii), $QN \parallel VU$

$\therefore \angle NPU = \angle QNP$ (alternate angles, $QN \parallel VU$)

but $\angle QPV = \angle NPU$ (reflection property)

$\therefore \angle QPV = \angle QNP$

$\therefore$ UV is a tangent to circle QPN (converse of alternate segment theorem)

Question 14

(a) (i) When $t = t_0$, H_1 will equal H_2 when $t = 2t_0$

i.e. $A + (B-A)e^{-k_2 \times 2t_0} = A + (B-A)e^{-k_1 t_0}$

$$e^{-2k_2 t_0} = e^{-k_1 t_0}$$

$$-2k_2 t_0 = -k_1 t_0$$

$$k_1 = 2k_2$$

(ii) Difference in temperatures $= H_2 - H_1$

$$= A + (B-A)e^{-k_2 t} - (A + (B-A)e^{-k_1 t})$$

$$= (B-A)(e^{-k_2 t} - e^{-k_1 t})$$

$$= (B-A)(e^{-k_2 t} - e^{-2k_2 t})$$

Method 1: Let $M = H_2 - H_1$

$$\frac{dM}{dt} = (B-A)(-k_2 e^{-k_2 t} + 2k_2 e^{-2k_2 t})$$

Let $\frac{dM}{dt} = 0$:

$$-\cancel{k_2} e^{-k_2 t} + 2\cancel{k_2} e^{-2k_2 t} = 0$$

$$\frac{2}{e^{2k_2 t}} = \frac{1}{e^{k_2 t}}$$

$$2 = e^{k_2 t}$$

$$k_2 t = \ln 2$$

$$t = \frac{1}{k_2}\ln 2$$

$$\frac{d^2M}{dt^2} = (B-A)\left(k_2^2 e^{-k_2 t} - 4k_2^2 e^{-2k_2 t}\right)$$

When $t = \frac{1}{k_2}\ln 2$:

$$\frac{d^2M}{dt^2} = (B-A) \times k_2^2\left(e^{-k_2 \cdot \frac{1}{k_2}\ln 2} - 4e^{-2k_2 \cdot \frac{1}{k_2}\ln 2}\right)$$

$$= (B-A) \cdot k_2^2 \left(e^{-\ln 2} - 4e^{-2\ln 2}\right)$$

$$= (B-A) \cdot k_2^2 \left(\frac{1}{2} - \frac{1}{4}\right)$$

$= (B-A) \cdot k_2^2 \times -\frac{1}{2}$

< 0 since $B > A$ and $k_2^2 > 0$.

$\therefore$ max. occurs when $t = \frac{1}{k_2} \ln 2$.

$\therefore$ Max. difference $= (B-A)\left(e^{-k_2 \times \frac{1}{k_2} \ln 2} - e^{-2k_2 \times \frac{1}{k_2} \ln 2}\right)$

$= (B-A)\left(\frac{1}{2} - \frac{1}{4}\right)$

$= \frac{B-A}{4}$ °C ✓

Method 2:

$H_2 - H_1 = (B-A)(e^{-k_2 t} - e^{-2k_2 t})$

let $x = e^{-k_2 t}$:

$H_2 - H_1 = (B-A)(x - x^2)$

$B-A$ is constant, and $y = x - x^2$ represents a concave down parabola, with local max. at

$x = -\frac{1}{2 \times -1}$, $y = \frac{1}{2} - (\frac{1}{2})^2$

$= \frac{1}{2}$, $= \frac{1}{2} - \frac{1}{4}$

$= \frac{1}{4}$ ✓

Check: $x = \frac{1}{2}$ gives $e^{-k_2 t} = \frac{1}{2}$

$-k_2 t = \ln(\frac{1}{2})$

$= -\ln 2$

$t = \frac{1}{k_2} \ln 2$ i.e. a valid result for t.

$\therefore$ Max. difference $= (B-A) \times \frac{1}{4}$

$= \frac{B-A}{4}$ °C ✓

(b) Step 1: $n=0$

$$\text{LHS} = \lim_{p\to\infty} \frac{p!}{(p-0)!\,p^0}$$

$$= \lim_{p\to\infty} \frac{p!}{p!}$$

$$= \lim_{p\to\infty} 1$$

$$= 1$$

$$= \text{RHS}$$

$\therefore$ true for $n=0$.

Step 2: Assume true for $n=k$.

i.e. $$\lim_{p\to\infty} \frac{p!}{(p-k)!\,p^k} = 1$$

Step 3: Prove true for $n=k+1$

To prove: $$\lim_{p\to\infty} \frac{p!}{(p-(k+1))!\,p^{k+1}} = 1$$

$$\lim_{p\to\infty} \frac{p!}{(p-k-1)!\,p^{k+1}} = 1$$

$$\text{LHS} = \lim_{p\to\infty} \frac{p!}{(p-k-1)!\,p^{k+1}}$$

$$= \lim_{p\to\infty} \frac{p!\,(p-k)}{(p-k)!\,p^{k+1}}$$

$$= \lim_{p\to\infty} \left[\frac{p!}{(p-k)!\,p^k} \times \frac{p-k}{p}\right]$$

$$= \lim_{p\to\infty} \frac{p!}{(p-k)!\,p^k} \times \lim_{p\to\infty} \frac{p-k}{p}$$ since both limits are defined.

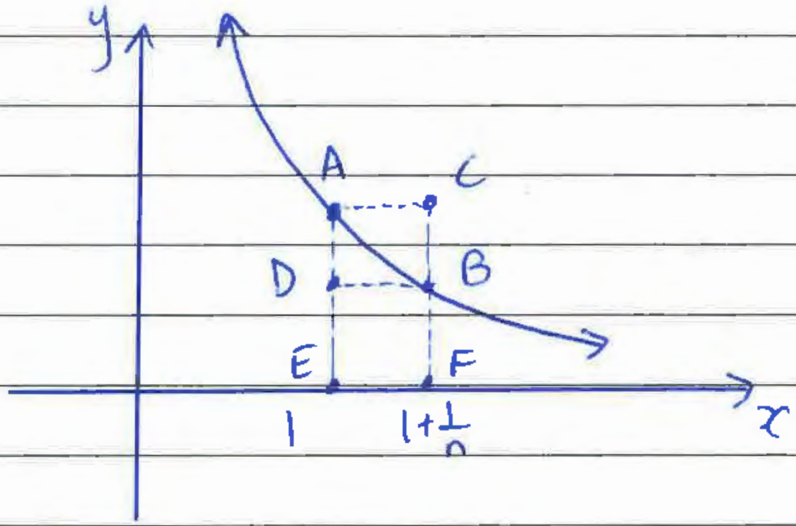
$$= 1 \times \lim_{p\to\infty} \left(1 - \frac{k}{p}\right)$$ (from assumption)

$$= 1 \times (1-0)$$

$$= 1$$

$$= \text{RHS}$$ $\therefore$ true for $n=k+1$, so true for integers $n \geq 0$ by mathematical induction.

(c)



(i) Area DBFE $\leq$ Area under curve $\leq$ Area AEFC

$$\frac{1}{n} \times \frac{1}{1+\frac{1}{n}} \leq \int_1^{1+\frac{1}{n}} \frac{dx}{x} \leq \frac{1}{n} \times 1$$

$$\therefore \frac{1}{n+1} \leq \int_1^{1+\frac{1}{n}} \frac{dx}{x} \leq \frac{1}{n}$$

(ii) $$\int_1^{1+\frac{1}{n}} \frac{dx}{x} = \left[\ln x\right]_1^{1+\frac{1}{n}}$$

$$= \ln\left(1+\frac{1}{n}\right) - \ln(1)$$

$$= \ln\left(1+\frac{1}{n}\right)$$

$$\therefore \frac{1}{n+1} \leq \ln\left(1+\frac{1}{n}\right) \leq \frac{1}{n}$$

$$e^{\frac{1}{n+1}} \leq e^{\ln\left(1+\frac{1}{n}\right)} \leq e^{\frac{1}{n}}$$

$$e^{\frac{n}{n+1}} \leq \left(1+\frac{1}{n}\right)^n \leq e$$

(iii) $$\lim_{n\to\infty} e^{\frac{n}{n+1}} = \lim_{n\to\infty} e^{\frac{1}{1+\frac{1}{n}}}$$

$$= e^{\frac{1}{1+0}}$$

$$= e$$

Since $e^{\frac{n}{n+1}} \to e$ from below, $\left(1+\frac{1}{n}\right)^n \to e$ also.

i.e. $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$

(d) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{n}\right)^k$ ✓ (recognising application of binomial theorem to $(1+\frac{1}{n})^n$ and writing correct expansion)

$$= \sum_{k=0}^{\infty}\lim_{n\to\infty}\binom{n}{k}\frac{1}{n^k}$$ since $0<\frac{1}{n}<1$ for $n>0$.

$$\therefore \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{\infty}\lim_{n\to\infty}\frac{n!}{(n-k)!\,k!\,n^k}$$

$$= \sum_{k=0}^{\infty}\lim_{n\to\infty}\frac{n!}{(n-k)!\,n^k}\times\frac{1}{k!}$$

From (b): $\lim_{p\to\infty}\frac{p!}{(p-n)!\,p^n} = 1$

Using dummy variables, $p \Rightarrow n$, $n \Rightarrow k$
$$\lim_{n\to\infty}\frac{n!}{(n-k)!\,n^k} = 1$$

$$\therefore \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{\infty} 1\times\frac{1}{k!}$$

Applying result from (c)(iii):
$$e = \sum_{k=0}^{\infty}\frac{1}{k!}$$
$$= \frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots$$
$$e = 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots$$ ✓